

## Exercise 8

Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when  $n = 1$ . Then, assuming that it is valid when  $n = m$  where  $m$  denotes any positive integer, show that it must hold when  $n = m + 1$ .

*Suggestion:* When  $n = m + 1$ , write

$$\begin{aligned}(z_1 + z_2)^{m+1} &= (z_1 + z_2)(z_1 + z_2)^m = (z_2 + z_1) \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m-k} \\ &= \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m+1-k} + \sum_{k=0}^m \binom{m}{k} z_1^{k+1} z_2^{m-k}\end{aligned}$$

and replace  $k$  by  $k - 1$  in the last sum here to obtain

$$(z_1 + z_2)^{m+1} = z_2^{m+1} + \sum_{k=1}^m \left[ \binom{m}{k} + \binom{m}{k-1} \right] z_1^k z_2^{m+1-k} + z_1^{m+1}.$$

Finally, show how the right-hand side here becomes

$$z_2^{m+1} + \sum_{k=1}^m \binom{m+1}{k} z_1^k z_2^{m+1-k} + z_1^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

### Solution

Use mathematical induction to prove the binomial theorem.

$$(z_1 + z_2)^n = \sum_{k=0}^n \binom{n}{k} z_1^k z_2^{n-k}, \quad n = 1, 2, \dots \quad (13)$$

Check that the formula is satisfied for the base case  $n = 1$ .

$$\begin{aligned}(z_1 + z_2)^1 &\stackrel{?}{=} \sum_{k=0}^1 \binom{1}{k} z_1^k z_2^{1-k} \\ z_1 + z_2 &\stackrel{?}{=} \sum_{k=0}^1 \frac{1!}{k!(1-k)!} z_1^k z_2^{1-k} \\ z_1 + z_2 &\stackrel{?}{=} \frac{1!}{0!1!} z_2 + \frac{1!}{1!0!} z_1 \\ z_1 + z_2 &= z_1 + z_2\end{aligned}$$

The base case is satisfied. Now assume the inductive hypothesis,

$$(z_1 + z_2)^m = \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m-k}. \quad (1)$$

The aim is to show

$$(z_1 + z_2)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

Proceed as suggested in the problem statement.

$$(z_1 + z_2)^{m+1} = (z_1 + z_2)(z_1 + z_2)^m$$

Use the inductive hypothesis, equation (1).

$$\begin{aligned} &= (z_1 + z_2) \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m-k} \\ &= z_1 \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m-k} + z_2 \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m-k} \\ &= \sum_{k=0}^m \binom{m}{k} z_1^{k+1} z_2^{m-k} + \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m+1-k} \end{aligned}$$

Replace  $k$  with  $k - 1$  in the first sum.

$$\begin{aligned} &= \sum_{k-1=0}^m \binom{m}{k-1} z_1^{(k-1)+1} z_2^{m-(k-1)} + \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m+1-k} \\ &= \sum_{k=1}^{m+1} \binom{m}{k-1} z_1^k z_2^{m+1-k} + \sum_{k=0}^m \binom{m}{k} z_1^k z_2^{m+1-k} \end{aligned}$$

Write out the last term of the first sum and the first term of the second sum.

$$\begin{aligned} &= \sum_{k=1}^m \binom{m}{k-1} z_1^k z_2^{m+1-k} + \binom{m}{m} z_1^{m+1} + \binom{m}{0} z_2^{m+1} + \sum_{k=1}^m \binom{m}{k} z_1^k z_2^{m+1-k} \\ &= \sum_{k=1}^m \left[ \binom{m}{k-1} z_1^k z_2^{m+1-k} + \binom{m}{k} z_1^k z_2^{m+1-k} \right] + \frac{m!}{m!0!} z_1^{m+1} + \frac{m!}{0!m!} z_2^{m+1} \\ &= \sum_{k=1}^m \left[ \binom{m}{k-1} + \binom{m}{k} \right] z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1} \\ &= \sum_{k=1}^m \left[ \frac{m!}{(k-1)!(m+1-k)!} + \frac{m!}{k!(m-k)!} \right] z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1} \\ &= \sum_{k=1}^m \left[ \frac{km!}{k(k-1)!(m+1-k)!} + \frac{(m+1-k)m!}{k!(m+1-k)(m-k)!} \right] z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1} \\ &= \sum_{k=1}^m \left[ \frac{km!}{k!(m+1-k)!} + \frac{(m+1-k)m!}{k!(m+1-k)!} \right] z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1} \\ &= \sum_{k=1}^m \left[ \frac{k\cancel{m!} + (m+1)m! - k\cancel{m!}}{k!(m+1-k)!} \right] z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1} \\ &= \sum_{k=1}^m \frac{(m+1)!}{k!(m+1-k)!} z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1} \\ &= \sum_{k=1}^m \binom{m+1}{k} z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} z_1^k z_2^{m+1-k} \end{aligned}$$

Therefore, by mathematical induction, the binomial theorem holds for complex numbers.